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On Study Recurrent Covariant Tensor Field of Second Order

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Abstract: **In this paper, we defined a $K^h -$ birecurrent space which is characterized by the condition $K^i_{jkh|m|l} = a_{lm}K^i_{jkh}$, $K^i_{jkh} \neq 0$, where a_{lm} is non – zero covariant tensor field of second order called *recurrence tensor*. The aim of this paper is to study the recurrence tensor field a_{lm} and to discuss the symmetric and skew – symmetric property of the recurrence covariant tensor field of second order a_{lm} in $K^h -$ birecurrent space, some results have been obtained . Also to introduced a $K^h -$ birecurrent affinely connected space , different identities concerning $K^h -$ birecurrent affinely connected space have been established .**

Keywords: **$K^h -$ birecurrent space, Recurrent covariant tensor field of second order, $K^h -$ birecurrent affinely connected space.**

1. INTRODUCTION

M.C. Chaki and A.N.Roychowdhary [4] have studied Ricci recurrent space of second order in Riemanni an geometry. B.B.Sinha and S.P.Singh [10] defined a Finsler space F_n for which the recurrent curvature tensor field K^i_{jkh} of second order satisfies the recurrence condition and studied the properties of the recurrence tensor field of second order a_{lm} in this space . H.D. Pande and S.D. Tripathi [5] discussed a Finsler space of the second order with the help of a symmetric non– zero recurrence tensor field, different theorems regarding it have obtained in an affinely connected space. P.N.Pandey [6] studied the recurrence vector field of a recurrent Finsler space when it is independent of the directional argument. F.Y.A.Qasem [7] discussed the recurrence tensor field of third order a_{nlm} , dealing with properties of the recurrence tensor of a normal projective trirecurrent Finsler space. F.Y.A.Qasem and M.A.A. Ali [8] studied the properties of $K^h -$ birecurrent affinely connected space .

Let F_n be an n – dimensional Finsler space equipped with the metric function F(x,y) satisfies the requisite conditions[9] .

É. Cartan ([2], [3]) deduced the h-covariant differatiation for an arbitrary vector field X^i with respect to x^k as follows :

(1.1) $X^i_{|k} \overset{\text{def}}{=} \partial_k X^i + X^r \Gamma^{*i}_{rk} - (\dot{\partial}_r X^i) G^r_k$.

The function Γ^{*i}_{rk} is defined by

(1.2) $\Gamma^{*i}_{rk} \coloneqq \Gamma^i_{rk} - C^i_{tr}\Gamma^t_{sk} y^s$.

The function Γ^{*i}_{rk} is Cartan's connection parameter, it is symmetric in its lower indices and positively homogeneous of degree zero in the directional argument y^i.

The functions Γ^{*i}_{rk} and G^r_k are related by

(1.3) a) $G^r_k = \Gamma^{*r}_{jk} y^s$,

Where

b) $G^r_k \coloneqq \dot{\partial}_k G^r$.

The function G^i is positively homogeneous of degree two in the directional argument y^i .

2. A RECURRENT COVARIANT TENSOR FIELD OF SECOND ORDER

A Finsler space for which Cartan's fourth curvature tensor K^i_{jkh} satisfies the birecurrence property with respect to Cartan's connection parameters Γ^{*i}_{kh} , i.e. characterized by the condition ([1] , [8])

(2.1) $K^i_{jkh|m|l} = a_{lm} K^i_{jkh}$, $K^i_{jkh} \neq 0$,

Where $|m|l$ is the h – covariant differential operator of the second order with respect to x^m and x^l , successively and a_{lm} is non – zero covariant tensor field of second order called *recurrence tensor field* .

Definition 2.1: A Finsler space F_n for which Cartan's fourth curvature tensor K^i_{jkh} satisfies the condition (2.1), will be called $K^h -$ *birecurrent space* , where a_{lm} is non – zero covariant tensor field of second order , the tensor satisfies the condition (2.1)

Will be celled $h-$ *birecurrent tensor* we shall denoted such space and tensor briefly by $K^h - BR - F_n$ and $h - BR$, respectively .

Let us consider an $K^h - BR - F_n$ which is characterized by the condition (2.1).

If we interchange the indices m and l in the condition (2.1) , we get

(2.2) $K^i_{jkh|m|l} - K^i_{jkh|l|m} = (a_{lm} - a_{ml})\, K^i_{jkh}$.

If the recurrence covariant tensor field of second order a_{lm} is symmetric , then the commutation formula (2.2) is vanished , i.e.

(2.3) $K^i_{jkh|m|l} - K^i_{jkh|l|m} = 0$.

Thus, we conclude

Theorem 2.1: *In an $K^h - BR - F_n$, the recurrence covariant tensor field of second order a_{lm} is non - symmetric .*

If the recurrence covariant tensor field of second order a_{lm} is skew - symmetric , then the equation (2.2) can be written as

(2.4) $K^i_{jkh|m|l} - K^i_{jkh|l|m} = 2a_{lm} K^i_{jkh}$.

Thus, we conclude

Theorem 2.2: *In an $K^h - BR - F_n$, if the recurrence covariant tensor field of second order a_{lm} is skew - symmetric , then the commutation formula for Cartan's second kind covariant differentiation is given by (2.4) .*

3. A RECURRENCE TENSOR IN K^h – BIRECURRENT AFFINELY CONNECTED SPACE

A Finsler space whose connection parameter G^i_{jk} is independent of the directional argument y^i is called *an affinely connected space (Berwald space)* .Thus , an affinely connected space characterized by any one of the following conditions

(3.1) a) $G^i_{jkh} = 0$ and b) $C_{ijk|h} = 0$.

The connection parameters Γ^{*i}_{jk} of Cartan and G^i_{jk} of Berwald coincide in affinely connected space and they are independent of the directional argument y^i [9] ,i.e.

(3.2) a) $G^i_{jkh} = \dot{\partial}_j G^i_{kh} = 0$ and b) $\dot{\partial}_j \Gamma^{*i}_{kh} = 0$.

Definition 3.1: The $K^h -$ birecurrent space which is an affinely connected space [satisfies any one of the conditions (3.1a) ,(3.1b) or (3.2b)] , will be called *a $K^h -$ birecurrent affinely connected space* and we shall denoted it briefly by $K^h - BR -$ *affinely connected space* .

Let us consider a $K^h - BR -$ affinely connected space.

The cyclic rotation of the indices l , m , k and h in the identity

(3.3) $a_{lm} K^i_{jkh} + a_{lh} K^i_{jmk} + a_{lk} K^i_{jhm} = 0$, {(2.2),[8]} , Gives

(3.4) $a_{mk}K^i_{jhl} + a_{ml}K^i_{jkh} + a_{mh}K^i_{jlk} = 0$,

(3.5) $a_{kh}K^i_{jlm} + a_{km}K^i_{jhl} + a_{kl}K^i_{jhm} = 0$

And

(3.6) $a_{hl}K^i_{jmk} + a_{hk}K^i_{jlm} + a_{hm}K^i_{jkl} = 0$.

Adding the four identities (3.3),(3.4),(3.5),(3.6) and using the skew – symmetric property of Cartan's fourth curvature tensor K^i_{jkh} in its last two lower indices , we get

(3.7) $(a_{lm} + a_{ml})\, K^i_{jkh} + (a_{lh} + a_{hl})\, K^i_{jmk} + (a_{lk} - a_{kl})\, K^i_{jkh}$

$+(a_{mk} + a_{km})\, K^i_{jhl} + (a_{mh} - a_{hm})\, K^i_{jkh} + (a_{kh} + a_{hk})\, K^i_{jlm} = 0$.

If the recurrence covariant tensor field of second order a_{rp} is symmetric , then the equation (3.7) can be written as

(3.8) $a_{lm}K^i_{jkh} + a_{lh}K^i_{jmk} + a_{mk}K^i_{jhl} + a_{kh}K^i_{jlm} = 0$.

Thus, we conclude

Theorem 3.1: *In $K^h - BR -$ affinely connected space , if the recurrence covariant tensor field of second order a_{rp} is symmetric , then the identity (3.8) holds good .*

If the recurrence covariant tensor field of second order a_{lp} is skew - symmetric , then the equation (3.7) can be written as

(3.9) $a_{lk}K^i_{jhm} - a_{hm}K^i_{jlk} = 0$.

Thus, we conclude

Theorem 3.2: *In $K^h - BR -$ affinely connected space , if the recurrence covariant tensor field of second order a_{rp} is skew - symmetric , then the identity (3.9) holds good .*

The cyclic rotation of the indices l , m , k and h in the identity

(3.10) $a_{lm}H^i_{jkh} + a_{lh}H^i_{jmk} + a_{lk}H^i_{jhm} = 0$, {(2.25),[8]} ,

Gives

(3.11) $a_{mk}H^i_{jhl} + a_{ml}H^i_{jkh} + a_{mh}H^i_{jlk} = 0$,

(3.12) $a_{kh}H^i_{jlm} + a_{km}H^i_{jhl} + a_{kl}H^i_{jhm} = 0$

And

(3.13) $a_{hl}H^i_{jmk} + a_{hk}H^i_{jlm} + a_{hm}H^i_{jkl} = 0$.

Adding the four identities (3.10),(3.11),(3.12),(3.13) and using the skew – symmetric property of Berwald's curvature tensor H^i_{jkh} in its last two lower indices , we get

(3.14) $(a_{lm} + a_{ml})\, H^i_{jkh} + (a_{lh} + a_{hl})\, H^i_{jmk} + (a_{lk} - a_{kl})\, H^i_{jkh}$

$+(a_{mk} + a_{km})\, H^i_{jhl} + (a_{mh} - a_{hm})\, H^i_{jkh} + (a_{kh} + a_{hk})\, H^i_{jlm} = 0$.

If the recurrence covariant tensor field of second order a_{rp} is symmetric , then the equation (3.14) can be written as

(3.15) $a_{lm}H^i_{jkh} + a_{lh}H^i_{jmk} + a_{mk}H^i_{jhl} + a_{kh}H^i_{jlm} = 0$.

Thus, we conclude

Theorem 3.3: *In $K^h - BR -$ affinely connected space , if the recurrence covariant tensor field of second order a_{rp} is symmetric , then the identity (3.15) holds good .*

If the recurrence covariant tensor field of second order a_{lp} is skew - symmetric , then the equation (3.14) can be written as

(3.16) $a_{lk}H^i_{jhm} - a_{hm}H^i_{jlk} = 0$.

Thus, we conclude

Theorem 3.4: *In $K^h - BR -$ affinely connected space , if the recurrence tensor field of second order a_{rp} is skew - symmetric , then the identity (3.16) holds good .*

The cyclic rotation of the indices l , m , k and h in the identity

(3.17) $a_{lm}H^i_{kh} + a_{lh}H^i_{mk} + a_{lk}H^i_{hm} = 0$, {(2.26),[8]} ,

Gives

(3.18) $a_{mk}H^i_{hl} + a_{ml}H^i_{kh} + a_{mh}H^i_{lk} = 0$,

(3.19) $a_{kh}H^i_{lm} + a_{km}H^i_{hl} + a_{kl}H^i_{hm} = 0$

And

(3.20) $a_{hl}H^i_{mk} + a_{hk}H^i_{lm} + a_{hm}H^i_{kl} = 0$.

Adding the four identities (3.17),(3.18),(3.19),(3.20) and using the skew – symmetric property of Berwald's curvature tensor H^i_{jkh} in its last two lower indices , we get

(3.21) $(a_{lm} + a_{ml})\, H^i_{jkh} + (a_{lh} + a_{hl})\, H^i_{jmk} + (a_{lk} - a_{kl})\, H^i_{jkh}$

$+(a_{mk} + a_{km})\, H^i_{jhl} + (a_{mh} - a_{hm})\, H^i_{jkh} + (a_{kh} + a_{hk})\, H^i_{jlm} = 0$

If the recurrence covariant tensor field of second order a_{rp} is symmetric , then the equation (3.21) can be written as

(3.22) $a_{lm}H^i_{jkh} + a_{lh}H^i_{jmk} + a_{mk}H^i_{jhl} + a_{kh}H^i_{jlm} = 0$.

Thus, we conclude

Theorem 3.5: *In $K^h - BR -$ affinely connected space , if the recurrence covariant tensor field of second order a_{rp} is symmetric , then the identity (3.22) holds good .*

If the recurrence covariant tensor field of second order a_{lp} is skew - symmetric , then the equation (3.21) can be written as

(3.23) $a_{lk}H^i_{jhm} - a_{hm}H^i_{jlk} = 0$.

Thus, we conclude

Theorem 3.6: *In $K^h - BR -$ affinely connected space , if the recurrence tensor field of second order a_{rp} is skew - symmetric , then the identity (3.23) holds good .*

The cyclic rotation of the indices l , m , k and h in the identity

(3.24) $a_{lk}R^r_{ijh} + a_{lh}R^r_{ikj} + a_{lj}R^r_{ihk} = 0$, {(2.28),[8]} ,

Gives

(3.25) $a_{mk}R^i_{jhl} + a_{ml}R^i_{jkh} + a_{mh}R^i_{jlk} = 0$,

(3.26) $a_{kh}R^i_{jlm} + a_{km}R^i_{jhl} + a_{kl}R^i_{jhm} = 0$

And

(3.27) $a_{hl}R^i_{jmk} + a_{hk}R^i_{jlm} + a_{hm}R^i_{jkl} = 0$.

Adding the four identities (3.24),(3.25),(3.26),(3.27) and using the skew – symmetric property of Berwald's curvature tensor H^i_{jkh} in its last two lower indices , we get

(3.28) $(a_{lm} + a_{ml})\, R^i_{jkh} + (a_{lh} + a_{hl})\, R^i_{jmk} + (a_{lk} - a_{kl})\, R^i_{jhm}$

$+(a_{mk} + a_{km})\, R^i_{jhl} + (a_{mh} - a_{hm})\, R^i_{jlk} + (a_{kh} + a_{hk})\, R^i_{jlm} = 0$.

If the recurrence covariant tensor field of second order a_{rp} is symmetric , then the equation (3.28) can be written as

(3.29) $a_{lm} R^i_{jkh} + a_{lh} R^i_{jmk} + a_{mk} R^i_{jhl} + a_{kh} R^i_{jlm} = 0$.

Thus, we conclude

Theorem 3.7: *In $K^h - BR -$ affinely connected space , if the recurrence covariant tensor field of second order a_{rp} is symmetric , then the identity (3.29) holds good .*

If the recurrence covariant tensor field of second order a_{lp} is skew - symmetric, then the equation (3.28) can be written as

(3.30) $a_{lk} R^i_{jhm} - a_{hm} R^i_{jlk} = 0$.

Thus, we conclude

Theorem 3.8: *In $K^h - BR -$ affinely connected space, if the recurrence tensor field of second order a_{rp} is skew - symmetric , then the identity (3.30) holds good .*

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